Differential Equation

① Solve $\sin^2 x \dfrac{d^2y}{dx^2} = 2y$ given that $y = \cot x$ is a solⁿ.

Solⁿ: →

We can writing the given eqⁿ in standard form we have

$$y'' + 0\cdot y' - 2\,\mathrm{cosec}^2 x \cdot y = 0 \longrightarrow ①$$

Comparing ① with $y'' + Py' + Qy = R$,

$$P = 0;\ Q = -2\,\mathrm{cosec}^2 x,\ R = 0 \longrightarrow ②$$

Given that $u = \cot x \longrightarrow ③$ is a part of C.F.

Let the general solⁿ of ① be

$$y = u\cdot v \longrightarrow ④$$

Then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left(0 + \frac{2}{\cot x}\frac{d\cot x}{dx}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} - \frac{2\,\mathrm{cosec}^2 x}{\cot x}\frac{dv}{dx} = 0 \longrightarrow ⑤$$

Let $\dfrac{dv}{dx} = q$ then $\dfrac{d^2v}{dx^2} = \dfrac{dq}{dx}$

$$\frac{dq}{dx} - \frac{4}{2\sin x\cos x}\,q = 0$$

$$\frac{dq}{dx} = 4\,\mathrm{cosec}\,2x \cdot q$$

$$\int \frac{dq}{q} = 4\int \mathrm{cosec}\,2x \cdot dx$$

$$\log q = 4\left(\tfrac{1}{2}\right)\log\tan x + \log c_1$$

$$q = c_1 \tan^2 x$$

$$\frac{dv}{dx} = c_1\tan^2 x \Rightarrow dv = c_1 \tan^2 x\, dx$$

$$dv = c_1(\sec^2 x - 1)\,dx \Rightarrow v = c_1(\tan x - x) + c_2 \longrightarrow ⑦$$

Then general solⁿ is

$$y = u\cdot v$$

$$\boxed{y = \cot x\,[c_1(\tan x - x) + c_2]}$$

Removal of the First Derivative
or
Normal Form

(2) Solve $$\frac{d^2y}{dx^2} - \frac{1}{\sqrt{x}}\frac{dy}{dx} + \frac{y}{4x^2}\left(-8+\sqrt{x}+x\right) = 0$$

Soln:→

Comparing the given eqn with $y'' + Py' + Qy = R$

Where
$$P = -\frac{1}{\sqrt{x}} \; ; \; Q = \frac{-8+\sqrt{x}+x}{4x^2} \;\&\; R = 0 \longrightarrow (1)$$

We choose
$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int -\frac{1}{\sqrt{x}}\,dx} = e^{+\frac{1}{2}\int x^{-\frac{1}{2}}\,dx} = e^{\frac{\cancel{1}}{\cancel{2}}\frac{x^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}} = e^{\sqrt{x}}$$

$$u = e^{\sqrt{x}} \longrightarrow (2)$$

Let the required general soln be $y = uv \longrightarrow (3)$

Then v is given by the normal form ~~$\frac{d^2y}{}$~~
$$\frac{d^2v}{dx^2} + Iv = S \longrightarrow (4)$$

Where
$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \left(-\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x}\right)$$
$$- \frac{1}{4}\left(-\frac{1}{\sqrt{x}}\right)^2 - \frac{1}{2}\left(+\frac{1}{2}\right)\frac{1}{x^{3/2}}$$
$$= \left(-\frac{2}{x^2} + \cancel{\frac{1}{4x^{3/2}}} + \cancel{\frac{1}{4x}}\right) - \cancel{\frac{1}{4x}} - \cancel{\frac{1}{4x^{3/2}}} = -\frac{2}{x^2}$$

$$S = \frac{R}{u} = 0$$

Then from eqn (4)
$$\frac{d^2v}{dx^2} - \frac{2}{x^2}v = 0$$

$$\left[x^2D^2 - 2\right]v = 0$$

Let $x = e^z \Rightarrow z = \log x$; $D = D_1$, $D^2 = D_1(D_1-1)$ $\{D_1 = \frac{d}{dz}\}$

$$\left[D_1(D_1-1) - 2\right]v = 0 \longrightarrow (5)$$

$$[D_1^2 - D_1 - 2]\, v = 0$$

A.E. is $\quad m^2 - m - 2 = 0$

$$m^2 - 2m + m - 2 = 0$$

$$m(m-2) + 1(m-2) = 0$$

$$(m-2)(m+1) = 0$$

$$m = -1, 2$$

Soln. $\quad v = c_1 e^{-z} + c_2 e^{2z}$

$\because \quad e^z = x$

$$v = \frac{c_1}{x} + c_2 x^2$$

Put in eqn (3)

$$y = u \cdot v$$

$$\boxed{y = e^{\sqrt{x}} \left[\frac{c_1}{x} + c_2 x^2 \right]}$$